Wavelet-induced renormalization group for the Landau-Ginzburg model

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The scale hierarchy of wavelets provides a natural frame for renormalization. Expanding the order parameter of the Landau-Ginzburg/Φ^4 model in a basis of compact orthonormal wavelets explicitly exhibits the coupling between scales that leads to non-trivial behavior. The locality properties of Daubechies' wavelets enable us to derive the qualitative renormalization flow of the Landau-Ginzburg model from Gaussian fluctuations in wavelet space.

1. INTRODUCTION

Daubechies' wavelets[1,2] are an orthonormal basis that explicitly separates scales. The basis functions are generated from a single set of functions $\psi_t(x)$ by dyadic dilatations and translations. Each basis function is expressed as

$$\psi_t^{(n)}(x')(x) = 2^{-nD/2}\psi_t(2^{-n}(x-x')) \quad , \qquad (1)$$

where $n \in \mathbb{Z}$ is the scale, $x' \in \mathcal{L}^n$ the position on a grid $\mathcal{L}^n = (2^n\mathbb{Z})^D$ with spacing 2^n, and $t = 1, \ldots, n_t = 2^D - 1$ determines how the D-dimensional wavelet is composed of one-dimensional functions. The resulting wavelet fields naturally live on lattices $\mathcal{L}^n$ of a dyadic multigrid.

It was proved only by Daubechies that a compact basis with these features (and good analytical properties) exists. It appears a natural basis to perform renormalization in. In this contribution, we demonstrate how a wavelet basis can be used to exhibit explicitly the interscale coupling introduced by the Φ^4 term and how an approximation to the renormalization flow in the Landau-Ginzburg/Φ^4 model can be derived from this.

2. WAVELET EXPANSION OF THE Φ^4 THEORY

We consider the Landau-Ginzburg/Φ^4 theory [3] with a one-component real order parameter $S(x)$, $x \in \mathbb{R}^D$, governed by the Hamiltonian

$$\mathcal{H} = \int \mathrm{d}^D x \left[\frac{1}{2}\left(\nabla S(x)\right)^2 + \frac{r_0}{2}S(x)^2 + \frac{u_0}{2}S(x)^4\right] (2)$$

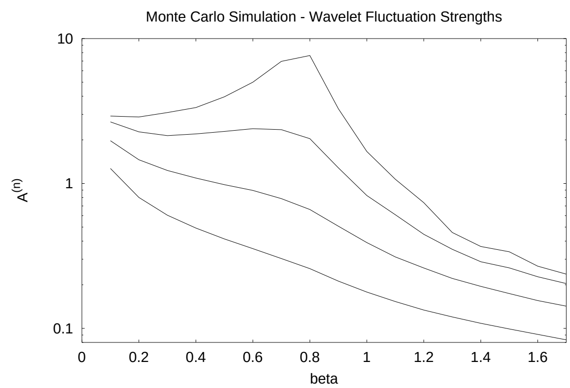

Figure 1. Wavelet fluctuation strengths at different scales as a function of the inverse temperature β, measured in a Monte Carlo simulation

with coupling constants r_0 and u_0. The field is expanded in a wavelet basis:

$$S(x) = \sum_n \sum_t^{n_t} \sum_{x' \in \mathcal{L}^n} S_t^{(n)}(x')\psi_t^{(n)}(x')(x) + S_0 \quad (3)$$

Since wavelets have vanishing first moments, we use the real number S_0 to represent the overall magnetization of the system, while the $S_t^{(n)}(x')$ represent fluctuations at different scales. Fig. 1 shows their strength $\left\langle \left[S_t^{(n)}(x')\right]^2 \right\rangle$ as measured in a Monte Carlo simulation. As they show very clearly the location of the phase transition, we will focus on modelling these quantities.

We thus make the variational ansatz that fluctuations are Gaussian and diagonal in wavelet

space:

$$\langle S_{t_1}^{(n_1)}(x'_1) S_{t_2}^{(n_2)}(x'_2)\rangle = \delta_{n_1,n_2}\delta_{t_1,t_2}\delta_{x'_1,x'_2}\,\mathcal{A}_{t_1}^{(n_1)} \quad (4)$$

Note that local fluctuations in wavelet space still provide for a nontrivial correlator in position space, as the decay of the correlator is encoded in the relative strength of fluctuations at different scales n.

The magnitude of the fluctuations $\mathcal{A}_t^{(n)}$ is then found by the variational method from minimizing the free energy $\mathcal{F} = U - \mathcal{S}/\beta$ where the internal energy U is the expectation value of the Hamiltonian H in the Gaussian ensemble (4), and $\mathcal{S}$ the entropy of this ensemble.

When calculating the internal energy, the self-similarity of the wavelet basis comes into play: As all basis functions are built from a single mother wavelet, the matrix elements of the Laplace operator have a simple scaling form:

$$\int \mathrm{d}^D x\, \psi_{t_1}^{(n_1)}(x'_1)(x)\Delta\psi_{t_2}^{(n_2)}(x'_1)(x) = 2^{-2n} C_{t_1,t_2} \quad . \quad (5)$$

Similarly, the four-point overlap integral occurring from the Φ^4 term has an approximate scaling representation. In this approximation, the effective internal energy per site reads simply

$$\begin{aligned} \frac{U}{N_0} \quad = \quad & \frac{1}{2}\sum_{nt} 2^{-n(D+2)}(-C_{tt})\,\mathcal{A}_t^{(n)} \\ & +\frac{r_0}{2}\mathcal{A} + \frac{3u_0}{2}\,\mathcal{A}^2 + 3u_0\bar{S}^2\mathcal{A} \\ & +\frac{r_0}{2}\bar{S}^2 + \frac{u_0}{2}\bar{S}^4 \end{aligned} \quad (6)$$

($\bar{S} = \langle S_0\rangle$) with the fluctuation sum

$$\mathcal{A} = \sum_n \sum_t 2^{-nD}\mathcal{A}_t^{(n)} \quad . \quad (7)$$

while the Gaussian terms are linear in the fluctuation strengths, the Φ^4 term introduces a coupling between scales. Since wavelets have compact support, the overlap integrals and thus the nonlinear contribution is finite, as opposed to the situtation with a Fourier basis. The canonical dimension 2 of the Laplace operator enters in the weight factor of the first term.

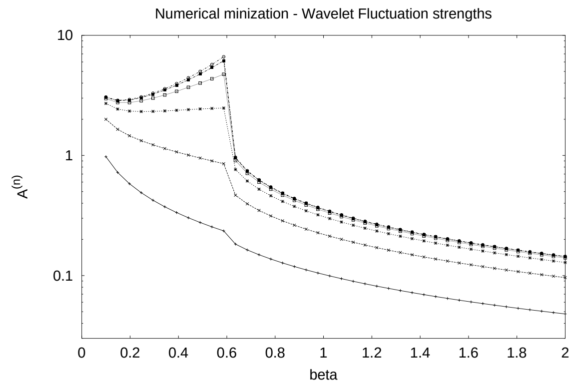

Figure 2. Wavelet fluctuation strengths at different scales as a function of the inverse temperature β, calculated from minimizing the effective internal energy.

Minimizing the free energy now with respect to the magnetization $\bar{S} = \langle S_0\rangle$ yields

$$\bar{S} = 0 \qquad \text{or} \qquad \bar{S} = \sqrt{-\frac{r_0}{2u_0} - 3\mathcal{A}} \quad . \quad (8)$$

Thus spontaneous symmetry breaking occurs when the fluctuation sum $\mathcal{A}$ exceeds the critical value $-r_0/6u_0$. Similarly, minimizing with respect to the $\mathcal{A}^{(n)}$ results in

$$\mathcal{A}_t^{(n)} = \frac{1}{\beta}\,\frac{1}{\frac{1}{2}\,2^{-2n}(-C_{tt}) + \frac{1}{2}r_0 + 3u_0(\mathcal{A}+\bar{S}^2)} \quad (9)$$

This is an implicit equation for the fluctuation sum $\mathcal{A}$ that can be solved numerically. Fig. 2 shows such a solution.

For $u_0 = 0$, (9) would be the wavelet representation of a exponentially decaying correlator with inverse correlation length $\sqrt{r_0}$. We can see that the interaction leads to a redefinition of the inverse correlation length depending on the quantity $\mathcal{A}$ which is defined implicitly by this equation. In particular, the correlation length diverges at the same point as the second solution in (8) becomes real, signifying the phase transition.

3. RENORMALIZATION FLOW

To derive the renormalization group flow of the theory, we apply the idea of removing the fine

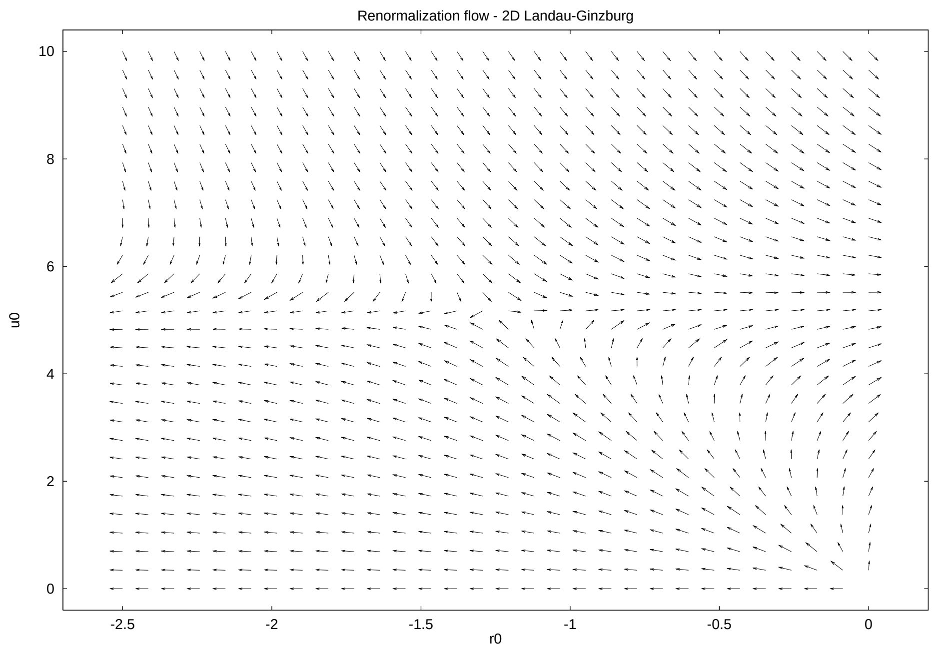

Figure 3. Renormalization flow of the effective Wavelet Hamiltonian in two dimensions. Shown is the $r_0 > 0$, $u_0 < 0$ quadrant of the coupling-constant plane. The arrows indicate the direction of the renormalization map. The Gaussian fixed point is at the lower right corner, the second fixed point at the center. To the right would be the symmetric phase, to the left the symmetry-broken phase.

degrees of freedom to the minimization of the free energy, i.e., we perform the minimization scale by scale.

Assume there exists a lowest scale $n = 0$, e.g. by already having eliminated all scales finer than n. We can then minimize the free energy with respect to $\mathcal{A}_t^{(0)}$, keeping all coarser scales as variables. $\mathcal{A}_t^{(0)}$ will then become a function of the $\mathcal{A}_t^{(n)}$, $n > 0$. By reinserting this function in the expression for the free energy and expanding in a Taylor series, one reaches a new effective free energy, now only depending on $\mathcal{A}_t^{(n)}$, $n > 0$, in which the new terms arising from the expansion can be absorbed (to some order) in a redefinition of the coupling constants. This yields a renormalization flow as a mapping $(r_0, u_0) \to (r_0', u_0')$ in the coupling constant plane. It turns out that there exists the Gaussian fixed point $r_0 = u_0 = 0$, and in $D < 4$ a second fixed point, corresponding to the Wilson-Fisher fixed point.

The actual position of the fixed points still depends on the matrix element of the Laplacian $-C_{tt}$ and thus on the type of wavelet. This shows a limitation of our approximation as we disregarded the correlation between neighboring wavelets which is basically determined by the extent of the wavelet.

4. CONCLUSIONS

A wavelet expansion can be used to derive the properties of the Landau-Ginzburg model and its nontrivial renormalization flow even in a rather simple approximation. The crucial features we have made use of are scaling and self-similarity of the basis and locality of the basis functions. They enabled us to focus on the fluctuation strengths at different scales as the quantities of interest that govern the phase transition. The effective free energy of the system exhibits in a minimal way the coupling between different scales.

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